

The monodromy matrix method of solving an exterior boundary value problem for a given stationary axisymmetric perfect fluid solution

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Abstract

A procedure is described for matching a given stationary axisymmetric perfect fluid solution to a not necessarily asymptotically flat vacuum exterior. Using data on the zero pressure surface, the procedure yields the Ernst potential of the matching vacuum metric on the symmetry axis. From this the full metric can be constructed using a variety of well established procedures.

1 Introduction

Because of the tremendous strides that were taken during the 1970's and 1980's in coping with the mathematical problems presented by stationary axisymmetric vacuum (SAV) and electrovac fields, the attention of many workers in this field has shifted in recent years to those even more challenging problems that are associated with stationary axisymmetric perfect fluid (SAPF) spacetimes. Except in the case of zero pressure fluids (dust) there exists no general method for solving the Einstein field equations within the

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fluid itself. Neither has any general method for joining a given SAPF solution to a vacuum SAV exterior at a zero pressure surface been described. It is the latter problem to which we shall propose a solution in this paper.

From the outset we stress that we are *not* insisting that the SAV metric be asymptotically flat. Ours is a method with which to effect a matching of a *given* SAPF to a SAV exterior, and not a method for solving a *global* problem. In the latter case, the asymptotic flatness and the singularity free nature of the solution are built in ab initio, and the SAPF metric as well as the SAV metric are determined a posteriori. The rotating dust disk solution of Neugebauer and Meinel[1] remains the only known global solution of the stationary axisymmetric field equations corresponding to a physically interesting rotating source, the analog of the classical zero pressure MacLaurin disk.

Restricting their attention to dust disks, Neugebauer and Meinel employed a method involving two integral equations, the “small” and “big” integral equations. The former integral equation was derived from the Neugebauer linear system for the Ernst equation under the assumption that on the disk $0 \leq \rho \leq \rho_0$ the Ernst potential $\mathcal{E}(0, \rho)$ is real and independent of ρ , and under the assumption that there exists a singularity free asymptotically flat matching SAV metric. The solution of this first integral equation yielded the axis values $\mathcal{E}(z, 0)$ for $z > 0$ of the Ernst potential. The second integral equation was derived from a Riemann-Hilbert problem, and was used in order to obtain the Ernst potential $\mathcal{E}(z, \rho)$ of the vacuum metric that matched the axis data derived in the first step. Finally, the surface density of the dust disk was determined by computing the jump in one of the metric fields as the disk is crossed.

In the present paper, and with an eye toward providing a procedure that can be employed routinely by those who discover new perfect fluid solutions, we develop the basic machinery that will be required to treat the “joining problem” for a broad class of spinning fluid spheroids that will be defined in Sec. 3C. Our approach, which is a variation of the “monodromy transform” approach of Alekseev[2], employs matching data, essentially certain metric components and their differentials, on the zero pressure surface to compute a 2×2 matrix function $\Pi(\tau)$ of a complex spectral parameter τ , the so-called “monodromy matrix.”

The axis values of the Ernst potential of the exterior SAV solution can be determined immediately from $\Pi(\tau)$, which means that any of the standard methods of generating a SAV solution from its axis data can be employed.

Thus, for example, the SAV solution can be constructed by solving a homogeneous Hilbert problem (HHP) that was developed by the authors[3, 4, 5] to effect Kinnersley-Chitre[6] transformations.

Since we are dealing with a group, there exist a number of sequences of transformations that can be used in principle to transform Minkowski space into the final SAV metric. Certain transformations may be found to be easier to carry out than others, so, here, as in many things, experience is the best teacher. However, if one wants a standardized treatment method, that would be obtained by employing an HKX transformation[7], either carried out as described by the authors of the HKX transformation, or by transforming Minkowski space into a static Weyl metric, a step that requires only a quadrature, then transforming that Weyl metric into the final SAV, a step that involves solving an ordinary Fredholm equation of the second kind[5]. Moreover, a possible alternative approach has been provided by Klein and Richter[8].

Even if the Π -matrix cannot be evaluated in closed form, by studying its domain of holomorphy in the extended τ -plane one can ascertain whether or not an asymptotically flat singularity free global solution exists. In Sec. 4 we shall illustrate the various steps of the Π -matrix method with two examples. The Schwarzschild interior solution is ridiculously simple but has the advantage of involving only short and easy calculations. The second example involves a rigidly rotating stationary axisymmetric dust spacetime with a dust density distribution that is specified a priori. There, since the pressure vanishes identically, one has the option of selecting at will the matching surface. After choosing a simple surface, we were able to show that the matching SAV metric is certainly not asymptotically flat. It would be exciting if one were to find a dust density distribution and a choice of matching surface that resulted in an asymptotically flat matching SAV metric.

Constructing a matching vacuum metric for the Wahlquist interior solution is likely to be challenging. In that case we do not expect the matching SAV spacetime to be asymptotically flat. We hope that the availability of the general procedure for matching a given fluid solution to a vacuum metric that we shall describe in this paper will encourage those who discover a new exact SAPF solution to investigate possible physical interpretations of their fluid solution, even if it turns out that the matching SAV metric is not asymptotically flat.

2 The Perfect Fluid Field Equations

In this section we shall formulate the field equations that will be assumed to hold both within the region occupied by a perfect fluid and within the vacuum region outside the source, and we shall stipulate our continuity-differentiability premises.

Let

$$h_{cd} := \mathbf{K}_c \cdot \mathbf{K}_d \quad (c, d \in \{3, 4\}), \quad (2.1)$$

where \mathbf{K}_3 and \mathbf{K}_4 denote the usual rotational and timelike Killing vector fields, respectively, whereupon $h_{33} > 0$ except on the axis, and $h_{44} < 0$ everywhere. Let $(\mathcal{M}^2, g^{(2)})$ denote any one of the usual two dimensional Riemannian subspaces¹ of the spacetime such that its Hausdorff space \mathcal{M}^2 is orthogonal to the Killing vector orbits, and its atlas is \mathbf{C}^∞ . All the differential forms that we employ in this paper will be on a two dimensional Hausdorff space \mathcal{M}^2 or on subspaces thereof. In particular, f , ω and P will denote the 0-forms with domain \mathcal{M}^2 and values

$$f(\mathbf{x}) = -h_{44}(\mathbf{x}), \quad \omega(\mathbf{x}) := h_{34}(\mathbf{x})/f(\mathbf{x}) \quad (2.2)$$

and

$$P(\mathbf{x}) := [h_{34}(\mathbf{x})^2 - h_{33}(\mathbf{x})h_{44}(\mathbf{x})]^{1/2} \geq 0 \quad (2.3)$$

for all $\mathbf{x} \in \mathcal{M}^2$. Furthermore, e_{12} will denote our choice of the unit 2-form on \mathcal{M}^2 , and \star will denote the duality operator for the differential forms on \mathcal{M}^2 such that

$$\star e_{12} = 1, \quad \star 1 = e_{12} \quad (2.4)$$

and, for all 1-forms v and w on \mathcal{M}^2 ,

$$(\star v)w = e_{12}(v \cdot w), \quad (2.5)$$

where 1 denotes the identity mapping on \mathcal{M}^2 , and we follow the practice of suppressing ‘ \wedge ’ in all exterior products and derivatives of differential forms. One readily shows that, for all 1-forms v and w on \mathcal{M}^2 ,

$$\star \star v = -v, \quad v(\star w) = w(\star v). \quad (2.6)$$

Note that²

$$P(\mathbf{x}) = 0 \text{ if and only if } \mathbf{x} \in \mathcal{M}_{ax}^1, \quad (2.7)$$

where \mathcal{M}_{ax}^1 denotes the axis subspace of \mathcal{M}^2 .

¹These are the surfaces of constant (φ, t) .

²This excludes event horizons from consideration.

A The Matching Premises

In this first exposition of the Π -matrix formalism, we shall restrict attention to those \mathcal{M}^2 , the topology of which is described by Fig. 1.³ We assume that the common boundary \mathcal{M}_0^1 of the fluid interior \mathcal{M}_{in}^2 and vacuum exterior \mathcal{M}_{ex}^2 has a \mathbf{C}^1 parametric equation

$$\mathbf{x} = \mathbf{x}(\lambda) \quad (0 \leq \lambda \leq \pi) \quad (2.8)$$

that defines a homeomorphism of $[0, \pi]$ onto \mathcal{M}_0^1 such that the “north” and “south” poles, where the zero pressure surface \mathcal{M}_0^1 and the axis \mathcal{M}_{ax}^1 intersect, are given by

$$\mathbf{n} = \mathbf{x}(0), \quad \mathbf{s} = \mathbf{x}(\pi). \quad (2.9)$$

We also adopt the following (often tacitly assumed) premises:

For all \mathbf{C}^n 1-forms v and w on \mathcal{M}^2 , $\star v$ and $v \cdot w$ are \mathbf{C}^1 if $n \geq 1$ and \mathbf{C}^0 if $n = 0$. Also, p is \mathbf{C}^0 , f and ω are \mathbf{C}^1 , while P is \mathbf{C}^2 and $dP(\mathbf{x}) \neq 0$ at all $\mathbf{x} \in \mathcal{M}^2$. Finally, there exists a continuous 0-form Λ such that $P\Lambda$ is \mathbf{C}^1 and $\omega = P^2\Lambda$. (2.10)

These premises include all of the matching conditions at the zero pressure surface \mathcal{M}_0^1 , and they are consistent with the field equations that we shall now specify.

B Four Field Equations and the Euler Equation

It is convenient to introduce the 0-forms

$$\Gamma := -\frac{1}{2} \ln(dP \cdot dP) \quad \text{and} \quad \psi := \frac{1}{2} \ln f \quad (2.11)$$

and the 1-forms

$$\xi := -P^{-1} f^2 \star d\omega := -f^2 \star [\Lambda dP + d(P\Lambda)] \quad (2.12)$$

and

$$\eta := P \star d\psi - \frac{1}{2} \omega \xi. \quad (2.13)$$

³This restriction rules out, for example, sources of a toroidal shape.

In view of our premises, Γ and ψ are \mathbf{C}^1 , and ξ and η are \mathbf{C}^0 . From the definition (2.11) of Γ ,

$$e_1 := e^\Gamma \star dP \text{ and } e_2 := e^\Gamma dP \quad (2.14)$$

constitute an orthonormal pair of 1-forms on \mathcal{M}^2 , and

$$e_{12} = e_1 e_2 = e^{2\Gamma} (\star dP) dP. \quad (2.15)$$

We shall also employ the 0-forms ξ_I , ξ_{II} , $d_I\psi$ and $d_{II}\psi$ for which

$$\xi = (\xi_I) \star dP + (\xi_{II}) dP \text{ and } d\psi = (d_I\psi) \star dP + (d_{II}\psi) dP. \quad (2.16)$$

The restrictions to the fluid interior \mathcal{M}_{in}^2 and the vacuum exterior \mathcal{M}_{ex}^2 of each of the differential forms on \mathcal{M}^2 that we have defined above will be denoted by the letter employed for the differential form with the affixed subscripts ‘*in*’ and ‘*ex*’, respectively.

We now introduce a function α that will be called the *boost form* on \mathcal{M}_{in}^2 . Let \mathbf{U} denote the fluid world velocity field on \mathcal{M}_{in}^4 , where we recall that $\mathbf{U} \cdot \mathbf{U} = -1$. Another timelike unit vector on \mathcal{M}_{in}^4 is the restriction to \mathcal{M}_{in}^4 of

$$\mathbf{e}_t := -\frac{1}{\sqrt{f}} \mathbf{K}_4. \quad (2.17)$$

It is easy to show that $|\mathbf{e}_t \cdot \mathbf{U}| \geq 1$ throughout \mathcal{M}_{in}^4 . We select the sign of \mathbf{K}_4 so that $\mathbf{e}_t \cdot \mathbf{U} \geq 1$ and let α denote the 0-form on \mathcal{M}_{in}^2 such that

$$\cosh \alpha(\mathbf{x}) := \mathbf{e}_t \cdot \mathbf{U}(\mathbf{x}), \quad \text{sgn } \alpha := \text{sgn } U^3 \quad (2.18)$$

for all $\mathbf{x} \in \mathcal{M}_{in}^2$. The boost form will be employed in our expressions for the field equations.

To ensure that the field equations are meaningful at all points of $\mathcal{M}_{in}^2 \cup \mathcal{M}_{ex}^2$ including points on \mathcal{M}_{ax}^1 , we adopt the following premises, which are not necessarily independent of one another:

$$\begin{aligned} &\text{For all } \mathbf{C}^\infty \text{ 1-forms } v \text{ and } w \text{ on } \mathcal{M}_{in}^2 \cup \mathcal{M}_{ex}^2, \star v \text{ and } v \cdot w \text{ are } \mathbf{C}^\infty. \\ &\text{Also, } f_{in}, f_{ex}, P_{in}, P_{ex}, \Lambda_{in}, \Lambda_{ex}, p_{in}, \epsilon_{in} \text{ and } P_{in}^{-1} \sinh \alpha \text{ are } \mathbf{C}^\infty. \end{aligned} \quad (2.19)$$

These premises are certainly stronger than necessary, but they enable us to avoid complications that would obscure this first exposition of the Π -matrix method.

With the above premises, all but one of the field equations are as follows:

$$d(\Gamma + \psi) - P \{ (\star d\psi) d_I \psi + (d\psi) d_{II} \psi + (2f)^{-2} [(\star \xi) \xi_I + (\xi) \xi_{II}] \} = -\kappa P p e^{2\Gamma} dP, \quad (2.20)$$

$$d \star dP = -2\kappa P p e_{12}, \quad (2.21)$$

$$d\xi = \kappa f(p + \epsilon)(\sinh 2\alpha) e_{12}, \quad (2.22)$$

$$d\eta = -\frac{1}{2}\kappa P[2p + (p + \epsilon) \cosh 2\alpha] e_{12}, \quad (2.23)$$

where $\kappa := 8\pi G/c^4$ in conventional metric units, and it is to be understood that the restriction of the right sides of Eqs. (2.22) and (2.23) to \mathcal{M}_{ex}^2 are identically zero. The contracted Bianchi identity in \mathcal{M}_{ex}^4 is identically satisfied, and in \mathcal{M}_{in}^4 it is equivalent to the following general relativistic Euler equation (in which we suppress the subscripts ‘in’):

$$dp + (p + \epsilon)[(\cosh 2\alpha)d\psi - (\cosh 2\alpha - 1)dP/2P - (\sinh 2\alpha)\star\xi/2f] = 0. \quad (2.24)$$

Note that the field equations (2.20) and (2.21) hold throughout \mathcal{M}^2 including \mathcal{M}_0^1 . From Eqs. (2.7) and (2.20), $\Gamma(\mathbf{x}) + \psi(\mathbf{x}) = \Gamma(\mathbf{n}) + \psi(\mathbf{n})$ at all $\mathbf{x} \in \mathcal{M}_{ax}^1$. As is well known, the condition

$$\Gamma(\mathbf{x}) + \psi(\mathbf{x}) = 0 \text{ at all } \mathbf{x} \in \mathcal{M}_{ax}^1 \quad (2.25)$$

is⁴ necessary and sufficient for g to be locally Minkowskian on \mathcal{M}_{ax}^2 ; i.e., to have no “conical singularities” on \mathcal{M}_{ax}^2 . Therefore, we shall henceforth assume that $\Gamma(\mathbf{n}) + \psi(\mathbf{n}) = 0$.

Unlike Eqs. (2.20) and (2.21), the field equations (2.22) and (2.23) are generally defined only in the domain $\mathcal{M}_{in}^2 \cup \mathcal{M}_{ex}^2$. This situation changes when $\epsilon(\mathbf{x}) = 0$ at all $\mathbf{x} \in \mathcal{M}_0^1$, but we shall not pursue that topic in this paper.

It is desirable to extend the domains of all of the field equations in \mathcal{M}_{in}^2 to $\bar{\mathcal{M}}_{in}^2 := \mathcal{M}_{in}^2 \cup \mathcal{M}_0^1$, of all of the field equations in \mathcal{M}_{ex}^2 to $\bar{\mathcal{M}}_{ex}^2 := \mathcal{M}_{ex}^2 \cup \mathcal{M}_0^1$, and of the Euler equation (2.24) to $\bar{\mathcal{M}}_{in}^2$. To accomplish this, it is sufficient⁵

⁴granted our premises

⁵It is to be understood that all derivatives at boundary points of $\bar{\mathcal{M}}_{in}^2$ and of $\bar{\mathcal{M}}_{ex}^2$ are defined using only sequences of points in \mathcal{M}_{in}^2 and \mathcal{M}_{ex}^2 , respectively.

to introduce the following reasonable premises, which are consistent with all other premises of this paper and with Eqs. (2.20) to (2.24):

$$\begin{aligned}
&\text{For all } \mathbf{C}^2 \text{ 1-forms } v \text{ and } w \text{ on } \bar{\mathcal{M}}_{in}^2, \star v \text{ and } v \cdot w \\
&\text{are } \mathbf{C}^2; \text{ and } P_{in}, f_{in}, \Lambda_{in} P_{in}, \Lambda_{in}, p_{in}, \epsilon_{in} \text{ and} \\
&P_{in}^{-1} \sinh \alpha \text{ have } \mathbf{C}^3, \mathbf{C}^2, \mathbf{C}^2, \mathbf{C}^1, \mathbf{C}^1, \mathbf{C}^0 \text{ and } \mathbf{C}^0 \\
&\text{extensions, respectively, to } \bar{\mathcal{M}}_{in}^2. \text{ For all } \mathbf{C}^\infty \text{ 1-forms} \\
&v \text{ and } w \text{ on } \bar{\mathcal{M}}_{ex}^2, \star v \text{ and } v \cdot w \text{ are } \mathbf{C}^\infty; \text{ and } P_{ex}, f_{ex} \\
&\text{and } \Lambda_{ex} \text{ have } \mathbf{C}^\infty \text{ extensions to } \bar{\mathcal{M}}_{ex}^2.
\end{aligned} \tag{2.26}$$

The only remaining field equation (one which we have chosen to suppress in this paper) happens to be a 2-form equation involving $d \star d\Gamma$. A well known proposition asserts that, if $(f_{ex}, P_{ex}, \Lambda_{ex})$ is a solution of Eqs. (2.21), (2.22) and (2.23) on $\bar{\mathcal{M}}_{ex}^2$, then⁶ Eq. (2.20) on $\bar{\mathcal{M}}_{ex}^2$ is completely integrable; and, if Γ_{ex} is the integral of Eq. (2.20) on $\bar{\mathcal{M}}_{ex}^2$, then $(f_{ex}, P_{ex}, \Lambda_{ex}, \Gamma_{ex})$ identically satisfies the $d \star d\Gamma$ field equation on $\bar{\mathcal{M}}_{ex}^2$. It can also be shown that if $(f_{in}, P_{in}, \Lambda_{in}, \Gamma_{in}, p_{in}, \epsilon_{in}, \alpha)$ is a solution of Eqs. (2.21), (2.22), (2.23), (2.24) and (suppressing the subscript ‘in’)

$$d_I(\Gamma + \psi) - 2P \{d_I \psi \, d_{II} \psi + (2f)^{-2} \xi_I \, \xi_{II}\} = 0 \tag{2.27}$$

on $\bar{\mathcal{M}}_{in}^2$, then⁷ Eq. (2.20) on $\bar{\mathcal{M}}_{in}^2$ is completely integrable, and, if Γ_{in} is the integral of Eq. (2.20) on $\bar{\mathcal{M}}_{in}^2$, then $(f_{in}, P_{in}, \Lambda_{in}, \Gamma_{in}, p_{in}, \epsilon_{in})$ identically satisfies the $d \star d\Gamma$ field equation on $\bar{\mathcal{M}}_{in}^2$.

C The Vacuum Region

We now focus attention on Eqs. (2.21), (2.22) and (2.23) on $\bar{\mathcal{M}}_{ex}^2$. Since $\bar{\mathcal{M}}_{ex}^2$ is simply connected, these field equations are equivalent to the statement⁸ that \mathbf{C}^∞ 0-forms Z , χ and ϕ exist on $\bar{\mathcal{M}}_{ex}^2$ such that

$$\star dP = dZ, \tag{2.28}$$

⁶Note that we are taking the liberty of employing the same notations for differential forms on $\bar{\mathcal{M}}_{in}^2$ and $\bar{\mathcal{M}}_{ex}^2$ as we do for their restrictions to \mathcal{M}_{in}^2 and \mathcal{M}_{ex}^2 , respectively. However, keep in mind that the differential forms that we are considering are \mathbf{C}^∞ in $\mathcal{M}_{in}^2 \cup \mathcal{M}_{ex}^2$, but are not necessarily \mathbf{C}^∞ on \mathcal{M}_0^1 .

⁷Ibid

⁸We are taking the liberty of suppressing the subscripts ‘ex’ in ‘ P_{ex} ’, ‘ ξ_{ex} ’ and ‘ η_{ex} ’. Similar abbreviating liberties will be employed in later equations, and we shall depend on the context to help the reader avoid confusion.

$$\xi = d\chi, \quad (2.29)$$

$$\eta = d\phi, \quad (2.30)$$

Equation (2.28) enables us to introduce the Weyl canonical chart $\mathbf{x} \rightarrow (z, \rho) := (Z(\mathbf{x}), P(\mathbf{x}))$ that maps \mathcal{M}_{ex}^2 onto

$$\bar{\mathcal{D}}_{ex} := \{(Z(\mathbf{x}), P(\mathbf{x})) : \mathbf{x} \in \bar{\mathcal{M}}_{ex}^2\}. \quad (2.31)$$

As is well known, Eqs. (2.29) and (2.30) then yield the following elliptic Ernst equation for the potential $\mathcal{E} := f + i\chi$ expressed as a function $\mathcal{E}(z, \rho) = f(z, \rho) + i\chi(z, \rho)$ of Weyl canonical coordinates:

$$\begin{aligned} f(z, \rho) \left\{ \frac{\partial}{\partial z} \left[\rho \frac{\partial \mathcal{E}(z, \rho)}{\partial z} \right] + \frac{\partial}{\partial \rho} \left[\rho \frac{\partial \mathcal{E}(z, \rho)}{\partial \rho} \right] \right\} \\ + \rho \left\{ \left(\frac{\partial \mathcal{E}(z, \rho)}{\partial z} \right)^2 + \left(\frac{\partial \mathcal{E}(z, \rho)}{\partial \rho} \right)^2 \right\} = 0 \end{aligned} \quad (2.32)$$

throughout $\bar{\mathcal{D}}_{ex}$. We shall describe a way to determine the axis values of this \mathcal{E} -potential from the matching data at the zero pressure surface. Many methods are known that permit the construction of $\mathcal{E}(z, \rho)$ from $\mathcal{E}(z, 0)$, assuming that suitable premises are satisfied. It is, however, not our purpose to rehash these well known procedures.

3 The Monodromy Matrix $\Pi(\tau)$

A Determination of the Matching Data

The curve

$$\mathcal{D}_0 := \{(Z(\mathbf{x}), P(\mathbf{x})) : \mathbf{x} \in \mathcal{M}_0^1\} \quad (3.1)$$

that represents the matching surface is given by the parametric equations

$$z(\lambda) := Z(\mathbf{x}(\lambda)), \quad \rho(\lambda) := P(\mathbf{x}(\lambda)) \quad (0 \leq \lambda \leq \pi), \quad (3.2)$$

and the matching data comprise $z(\lambda)$, $\rho(\lambda)$ and

$$\begin{aligned} f(\lambda) &:= f(\mathbf{x}(\lambda)), \quad \chi(\lambda) := \chi(\mathbf{x}(\lambda)), \\ \omega(\lambda) &:= \omega(\mathbf{x}(\lambda)), \quad \phi(\lambda) := \phi(\mathbf{x}(\lambda)). \end{aligned} \quad (3.3)$$

The functions of λ that are defined above are to be determined from the given fluid solution in $\bar{\mathcal{M}}_{in}^2$ as follows.

The given fluid solution is a compact two-dimensional Riemannian space $(\bar{\mathcal{M}}^2, g_{in}^{(2)})$ with a maximal \mathbf{C}^∞ atlas. If (x^1, x^2) denotes the coordinate pair corresponding to any point \mathbf{x} in the domain of a chart in this atlas, one can use the metric $g_{in}^{(2)}$ to compute the 0-forms \mathcal{Y}_j^i for which

$$\star dx^i = dx^j \mathcal{Y}_j^i(\mathbf{x}). \quad (3.4)$$

Then we may write

$$(\star dP)(\mathbf{x}) = dx^j \mathcal{Y}_j^i(\mathbf{x}) P_i(\mathbf{x}), \quad (3.5)$$

where $P_i(\mathbf{x})$ is defined by

$$dP(\mathbf{x}) = dx^i P_i(\mathbf{x}). \quad (3.6)$$

The pull-back of $\star dP$ corresponding to the mapping $\lambda \rightarrow \mathbf{x}(\lambda)$ of $[0, \pi]$ into $\bar{\mathcal{M}}_{in}^2$ is, therefore, given by

$$dz(\lambda) := d\lambda \dot{x}^i(\lambda) \mathcal{Y}_i^j(\mathbf{x}(\lambda)) P_j(\mathbf{x}(\lambda)) \quad (3.7)$$

for each chart whose domain contains an interval of \mathcal{M}_0^1 , where $x^1(\lambda)$ and $x^2(\lambda)$ denote the coordinates of the point $\mathbf{x}(\lambda)$ in the interval. Integration of $dz(\lambda)$ now yields $z(\lambda)$ up to an additive constant $z(0)$ whose value can be chosen freely to fit some convention. On the other hand,

$$\rho(\lambda) := P(\mathbf{x}(\lambda)). \quad (3.8)$$

The pairs $(\chi(\lambda), \omega(\lambda))$ and $(\phi(\lambda), f(\lambda))$ are treated similarly to how we have here treated the pair $(z(\lambda), \rho(\lambda))$. By specifying all six of these objects, we are, in effect, specifying both the tangential and normal derivatives of $\rho(\mathbf{x})$, $f(\mathbf{x})$ and $\omega(\mathbf{x})$ on the zero pressure surface. Moreover, nothing is assumed concerning behavior of the solution at large distances from the fluid source. What we are solving then is neither a Dirichlet nor a Neumann problem.

B A Linear System for the Ernst Equation

In a recent paper[9] we described, and showed the relationships among, three linear systems for the vacuum Ernst equation, one inferred by the present authors from the Kinnersley-Chitre formalism, one developed and used extensively by Neugebauer and a new one that the present authors have found

valuable in the course of developing formal proofs, especially in connection with the hyperbolic Ernst equation.⁹ It is the last named one that we plan also to use here. It will be expressed in the form

$$d\mathcal{Q}(\mathbf{x}, \tau) = \Delta(\mathbf{x}, \tau)\mathcal{Q}(\mathbf{x}, \tau) \quad (3.9)$$

for all $\mathbf{x} \in \bar{\mathcal{M}}_{ex}^2$ and $\tau \in C - \bar{K}(\mathbf{x})$, where it is understood that d does not operate on the complex spectral parameter τ (i.e., $d\tau = 0$) and where

$$\begin{aligned} \Delta(\mathbf{x}, \tau) := & - \left(\frac{\tau - Z(\mathbf{x}) + P(\mathbf{x})\star}{\mu(\mathbf{x}, \tau)} \right) \left(\frac{Idf(\mathbf{x}) - Jd\chi(\mathbf{x})}{2f(\mathbf{x})} \right) \sigma_3 \\ & - J \frac{d\chi(\mathbf{x})}{2f(\mathbf{x})}, \end{aligned} \quad (3.10)$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3.11)$$

$$\mu(\mathbf{x}, \tau) := [(\tau - Z(\mathbf{x}))^2 + P(\mathbf{x})^2]^{1/2}, \quad \lim_{\tau \rightarrow \infty} \frac{\mu(\mathbf{x}, \tau)}{\tau} = 1, \quad (3.12)$$

and the cut $\bar{K}(\mathbf{x})$ in the complex τ -plane is a simple \mathbf{C}^1 arc whose end points are the branch points $Z(\mathbf{x}) \pm iP(\mathbf{x})$ of $\mu(\mathbf{x}, \tau)$. Moreover, $\bar{K}(\mathbf{x})$ is symmetric with respect to the real axis, is a subset of¹⁰

$$\bar{\Sigma}_{ex} := \{z \pm i\rho : (z, \rho) \in \bar{\mathcal{D}}_{ex}\} \quad (3.13)$$

and intercepts the real axis at a point on the same side of the closed contour

$$\Sigma_0 := \{z(\lambda) \pm i\rho(\lambda) : 0 \leq \lambda \leq \pi\} \quad (3.14)$$

as the point $z(0)$ that represents the north pole \mathbf{n} , i.e., the point of interception is in

$$\Sigma_{ax+} := \{z \in R^1 : (z, 0) \in \bar{\mathcal{D}}_{ex} \text{ and } z \geq z(0)\}. \quad (3.15)$$

The solution of Eq. (3.9) will be made unique by specifying that

$$\mathcal{Q}(\mathbf{n}, \tau) := e^{-\sigma_3 \psi(\mathbf{n})} \begin{pmatrix} 1 & -\chi(\mathbf{n}) \\ 0 & 1 \end{pmatrix} \quad (3.16)$$

⁹In the present paper our $\Delta(\mathbf{x}, \tau)$ and $\mathcal{Q}(\mathbf{x}, \tau)$ are those fields that were denoted by $\Gamma_{HE}(\mathbf{x}, \tau)$ and $\mathcal{F}_{HE}(\mathbf{x}, \tau)$, respectively, in the cited reference.

¹⁰See Eq. (2.31).

for all $\tau \in C$. We reserve the option of scaling $x^4 = t$ and choosing the arbitrary additive real constant in χ so that $\mathcal{E}(\mathbf{n}) = 1$, whereupon $\mathcal{Q}(\mathbf{n}, \tau) = I$. To grasp the motivation behind the selection (3.16), note that the Ernst equation (2.32) implies that $\partial\mathcal{E}(z, \rho)/\partial\rho$ vanishes at $\rho = 0$. Therefore, if one expresses all differential forms in Eqs. (3.9) and (3.10) as functions of (z, ρ, τ) ,

$$\Delta(z, 0, \tau) = -\sigma_3 d\psi(z, 0) - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e^{-2\psi(z, 0)} d\chi(z, 0). \quad (3.17)$$

Therefore, the solution on $\{(z, 0) \in \bar{\mathcal{D}}_{ex} : z \geq z(0)\}$ of Eqs. (3.9) and (3.16) is

$$\mathcal{Q}(z, 0, \tau) = e^{-\sigma_3 \psi(z, 0)} \begin{pmatrix} 1 & -\chi(z, 0) \\ 0 & 1 \end{pmatrix} \text{ for all } z \in \Sigma_{ax+} \text{ and } \tau \in C. \quad (3.18)$$

Three key properties of $\mathcal{Q}(\mathbf{x}, \tau)$ are easily deducible from Eqs. (3.9) and (3.16). First, since $\mathcal{Q}(\mathbf{n}, \tau)$ is real and $\Delta(\mathbf{x}, \tau) := \Delta(\mathbf{x}, \tau^*)^* = \Delta(\mathbf{x}, \tau)$, the reality condition

$$\mathcal{Q}^*(\mathbf{x}, \tau) := [\mathcal{Q}(\mathbf{x}, \tau^*)]^* = \mathcal{Q}(\mathbf{x}, \tau) \quad (3.19)$$

holds. Second, since $\text{tr } \Delta(\mathbf{x}, \tau) = 0$ and $\det \mathcal{Q}(\mathbf{n}, \tau) = 1$,

$$\det \mathcal{Q}(\mathbf{x}, \tau) = 1. \quad (3.20)$$

Third, for fixed $\mathbf{x} \in \bar{\mathcal{M}}_{ex}^2$, $\mathcal{Q}(\mathbf{x}, \tau)$ is a holomorphic function of τ throughout $C - \bar{K}(\mathbf{x})$.

Employing Eqs. (2.12), (2.13), (2.29), (2.30) and (3.10), one can express $\Delta(\mathbf{x}, \tau)$ in the following form that no longer contains the duality operator:

$$\begin{aligned} \Delta(\mathbf{x}, \tau) &:= -\frac{\tau - Z(\mathbf{x})}{\mu(\mathbf{x}, \tau)} \left(\frac{Idf(\mathbf{x}) - Jd\chi(\mathbf{x})}{2f(\mathbf{x})} \right) \sigma_3 \\ &\quad - \frac{1}{\mu(\mathbf{x}, \tau)} \left[I \left(d\phi(\mathbf{x}) + \frac{1}{2}\omega(\mathbf{x})d\chi(\mathbf{x}) \right) - \frac{1}{2}Jf(\mathbf{x})d\omega(\mathbf{x}) \right] \sigma_3 \\ &\quad - J \frac{d\chi(\mathbf{x})}{2f(\mathbf{x})}. \end{aligned} \quad (3.21)$$

Thus, we conclude that

$$\frac{\partial \mathcal{Q}(\mathbf{x}(\lambda), \tau)}{\partial \lambda} = \left\{ -\frac{\tau - z(\lambda)}{\mu(\mathbf{x}(\lambda), \tau)} \left(I \frac{\dot{f}(\lambda)}{2f(\lambda)} - J \frac{\dot{\chi}(\lambda)}{2f(\lambda)} \right) \sigma_3 \right.$$

$$\begin{aligned}
& - \frac{1}{\mu(\mathbf{x}(\lambda), \tau)} \left[I \left(\dot{\phi}(\lambda) + \frac{1}{2} \omega(\lambda) \dot{\chi}(\lambda) \right) \right. \\
& \left. - \frac{1}{2} J f(\lambda) \dot{\omega}(\lambda) \right] \sigma_3 \\
& - J \frac{\dot{\chi}(\lambda)}{2f(\lambda)} \Big\} \mathcal{Q}(\mathbf{x}(\lambda), \tau) \text{ for all } 0 \leq \lambda \leq \pi \\
& \text{and } \tau \in C - \bar{K}(\mathbf{x}(\lambda)), \tag{3.22}
\end{aligned}$$

where $\mathcal{Q}(\mathbf{x}(0), \tau) = \mathcal{Q}(\mathbf{n}, \tau)$ is given by Eq. (3.16), where dots denote derivatives with respect to λ , and where $\bar{K}(\mathbf{x}(\lambda))$ lies on Σ_0 and is

$$\bar{K}(\mathbf{x}(\lambda)) = \{z(\lambda') \pm i\rho(\lambda') : 0 \leq \lambda' \leq \lambda\}. \tag{3.23}$$

We note that only the matching data (3.2) and (3.3) are needed in order to be able to write out this ordinary differential equation. We shall employ the convenient abbreviation

$$\mathcal{Q}(\lambda, \tau) := \mathcal{Q}(\mathbf{x}(\lambda), \tau). \tag{3.24}$$

Thus, both the closed contour Σ_0 and the function $\mathcal{Q}(\lambda, \tau)$ are uniquely determined by the matching data $z(\lambda)$, $\rho(\lambda)$, $\chi(\lambda)$, $\omega(\lambda)$, $\phi(\lambda)$ and $f(\lambda)$, the differential equation (3.22) and the initial condition

$$\mathcal{Q}_0 := \mathcal{Q}(0, \tau) := e^{\sigma_3 \psi(0)} \begin{pmatrix} 1 & -\chi(0) \\ 0 & 1 \end{pmatrix}. \tag{3.25}$$

C Determination of the Π -Matrix

It is from the function $\mathcal{Q}(\lambda, \tau)$ that one computes the monodromy matrix $\Pi(\tau)$ provided that the latter exists. A class of spinning fluid spheroids for which monodromy matrices exist will now be defined. Specifically, we shall henceforth consider the class of all SAPF solutions $(\bar{\mathcal{M}}_{in}^2, g_{in}^{(2)})$ for which the following two sets of conditions hold:

- (i) The parts of the premises (2.19) and (2.26) that concern \mathcal{M}_{in}^2 and $\bar{\mathcal{M}}_{in}^2$, respectively, are satisfied. Also, $\dot{z}(\lambda)^2 + \dot{\rho}(\lambda)^2 > 0$ for all $\lambda \in [0, \pi]$.
- (ii) As regards the function \mathcal{Q} , there exist 2×2 matrix functions \mathcal{Q}_1 and \mathcal{Q}_2 with a common domain $[0, \pi] \times \hat{\Sigma}$ such that $\hat{\Sigma} \subset C$ and is an open

covering of Σ_0 that is either topologically equivalent to an annulus or is a simply connected open neighborhood of ∞ , $\hat{\Sigma}^* = \hat{\Sigma}$,

$$\begin{aligned} \mathcal{Q}(\lambda, \tau) &= \mathcal{Q}_1(\lambda, \tau) + \mu(\mathbf{x}(\lambda), \tau) \mathcal{Q}_2(\lambda, \tau) \\ \text{for all } \lambda &\in [0, \pi] \text{ and } \tau \in \hat{\Sigma} - \bar{K}(\mathbf{x}(\lambda)) \end{aligned} \quad (3.26)$$

and, for each $\tau \in \hat{\Sigma}$, $\mathcal{Q}_1(\lambda, \tau)$ and $\mathcal{Q}_2(\lambda, \tau)$ are \mathbf{C}^1 functions of λ throughout $[0, \pi]$. Also, for each $\lambda \in [0, \pi]$, $\mathcal{Q}_i(\lambda, \tau)$ and $\partial \mathcal{Q}_i(\lambda, \tau) / \partial \lambda$ ($i = 1, 2$) are holomorphic functions of τ throughout $\hat{\Sigma}$.

That completes the two sets of conditions. Conditions (ii) are required for the existence of the monodromy matrix. To illustrate how the conditions (ii) are employed in proofs, consider the fact that $\mathcal{Q}^*(\lambda, \tau) = \mathcal{Q}(\lambda, \tau)$ for all $\lambda \in [0, \pi]$ and $\tau \in C - \bar{K}(\mathbf{x}(\lambda))$.¹¹ Therefore, from Eq. (3.26),

$$\begin{aligned} [\mathcal{Q}_1^*(\lambda, \tau) - \mathcal{Q}_1(\lambda, \tau)] + \mu(\mathbf{x}(\lambda), \tau) [\mathcal{Q}_2^*(\lambda, \tau) - \mathcal{Q}_2(\lambda, \tau)] &= 0 \\ \text{for all } \lambda &\in [0, \pi] \text{ and } \tau \in \hat{\Sigma} - \bar{K}(\mathbf{x}(\lambda)). \end{aligned} \quad (3.27)$$

For each $0 < \lambda < \pi$, analytic continuation on any simple closed orbit that lies in $\hat{\Sigma}$ and that encloses one and only one of the branch points $z(\lambda) \pm i\rho(\lambda)$ of $\mu(\mathbf{x}(\lambda), \tau)$ exists and induces the replacements $\mathcal{Q}_i(\lambda, \tau) \rightarrow \mathcal{Q}_i(\lambda, \tau)$ and $\mu(\mathbf{x}(\lambda), \tau) \rightarrow -\mu(\mathbf{x}(\lambda), \tau)$ in Eq. (3.27), whereupon one obtains $\mathcal{Q}_i^*(\lambda, \tau) = \mathcal{Q}_i(\lambda, \tau)$ for all $i \in \{1, 2\}$, $0 < \lambda < \pi$ and $\tau \in \hat{\Sigma} - \bar{K}(\mathbf{x}(\lambda))$. The continuity properties of \mathcal{Q}_i then yields

$$\mathcal{Q}_i^*(\lambda, \tau) = \mathcal{Q}_i(\lambda, \tau) \text{ for all } i \in \{1, 2\}, \lambda \in [0, \pi] \text{ and } \tau \in \hat{\Sigma}. \quad (3.28)$$

For each $0 < \lambda < \pi$, analytic continuation of (3.26) on any simple closed orbit in $\hat{\Sigma}$ that encloses exactly one of the points $z(\lambda) \pm i\rho(\lambda)$ is seen to exist and yields

$$\mathcal{Q}'(\lambda, \tau) := \mathcal{Q}_1(\lambda, \tau) - \mu(\mathbf{x}(\lambda), \tau) \mathcal{Q}_2(\lambda, \tau). \quad (3.29)$$

Then, by letting $\mathcal{Q}'_i(0, \tau)$ and $\mathcal{Q}'_i(\pi, \tau)$ denote the limits of $\mathcal{Q}'(\lambda, \tau)$ as $\lambda \rightarrow 0$ and $\lambda \rightarrow \pi$, respectively, we extend Eq. (3.29) to all $\lambda \in [0, \pi]$ and $\tau \in \bar{K}(\mathbf{x}(\lambda))$. Since¹² $\det \mathcal{Q}(\lambda, \tau) = 1$, it follows by employing analytic continuation and the continuity properties of \mathcal{Q}_i in the manner illustrated by the proof of Eq. (3.28) that

$$\det \mathcal{Q}'(\lambda, \tau) = \det \mathcal{Q}(\lambda, \tau) = 1 \text{ for all } \lambda \in [0, \pi] \text{ and } \tau \in \hat{\Sigma} - \bar{K}(\mathbf{x}(\lambda)). \quad (3.30)$$

¹¹The derivation of this fact is similar to that of Eq. (3.19).

¹²See Eq. (3.20).

Furthermore, by employing analytic continuation together with the continuities as functions of λ of $\mathcal{Q}_i(\lambda, \tau)$ and $\partial \mathcal{Q}_i(\lambda, \tau)/\partial \lambda$, one proves that $-J\mathcal{Q}'(\lambda, \tau)J$ satisfies the same differential equation (3.22) as $\mathcal{Q}(\lambda, \tau)$ for all $\lambda \in [0, \pi]$ and $\tau \in \hat{\Sigma} - \bar{K}(\mathbf{x}(\lambda))$. Thus, $\Pi(\tau)$ exists for each $\tau \in \hat{\Sigma}$ such that

$$-J\mathcal{Q}'(\lambda, \tau)J = \mathcal{Q}(\lambda, \tau)\Pi(\tau) \quad (3.31)$$

for all $\lambda \in [0, \pi]$ and $\tau \in \hat{\Sigma} - \bar{K}(\mathbf{x}(\lambda))$. By then employing Eq. (3.30) and the fact that $M^T J M = (\det M)J$ for any 2×2 matrix M , one obtains

$$\Pi(\tau) = -J\mathcal{Q}(\lambda, \tau)^T \mathcal{Q}'(\lambda, \tau)J. \quad (3.32)$$

Therefore, with the aid of Eqs. (3.26) and (3.29), we obtain the convenient formula

$$\Pi(\tau) = -J\mathcal{Q}_0^T \mathcal{Q}'(0, \tau)J = -J\mathcal{Q}_0^T [2\mathcal{Q}_1(0, \tau) - \mathcal{Q}_0], \quad (3.33)$$

from which one sees that $\Pi(\tau)$ is holomorphic throughout $\hat{\Sigma}$.

From Eqs. (3.30) and (3.31) the determinantal property

$$\det \Pi(\tau) = 1 \quad (3.34)$$

follows, while from Eqs. (3.25), (3.28) and (3.33) the reality property

$$\Pi^*(\tau) = \Pi(\tau) \quad (3.35)$$

follows. Moreover, from Eq. (3.32) and the fact that analytic continuation on a simple closed orbit that encloses one and only one of the branch points $z(\lambda) \pm i\rho(\lambda)$ induces $\mathcal{Q}(\lambda, \tau) \rightarrow \mathcal{Q}'(\lambda, \tau)$ and $\mathcal{Q}'(\lambda, \tau) \rightarrow \mathcal{Q}(\lambda, \tau)$ the symmetry property

$$\Pi(\tau)^T = \Pi(\tau) \quad (3.36)$$

follows.

Upon letting $\lambda = 0$ and $\tau = z(0)$ in Eqs. (3.26) and then using Eq. (3.25), we obtain $\mathcal{Q}_1(0, z(0)) = \mathcal{Q}_0$. Therefore, from Eq. (3.33),

$$\Pi(z(0)) = -J\mathcal{Q}_0^T \mathcal{Q}_0 J. \quad (3.37)$$

It is clear from Eqs. (3.34) to (3.37) that

$$\begin{aligned} &\Pi(\tau) \text{ is unimodular, real, symmetric and} \\ &\text{positive definite for all } \tau \text{ in the maximal} \\ &\text{real subinterval of the real axis in } \hat{\Sigma} \\ &\text{that contains } z(0). \end{aligned} \quad (3.38)$$

D Use of $\Pi(\tau)$ to Assess Asymptotic Flatness

It is unnecessary to construct the exterior vacuum solution in order to determine if that solution is asymptotically flat, for this can be determined by investigating instead the analytic properties of $\Pi(\tau)$. To help describe this investigation, we shall need the following concepts:

$$\begin{aligned} \Sigma_{EXT} &:= \text{that open unbounded subset of } C \\ &\quad \text{whose boundary is } \Sigma_0, \end{aligned} \quad (3.39)$$

$$\bar{\Sigma}_{EXT} := \Sigma_{EXT} \cup \Sigma_0, \quad (3.40)$$

$$\bar{S}_{EXT} := \{(z, \rho) \in R^2 : (z + i\rho) \in \bar{\Sigma}_{EXT}\}, \quad (3.41)$$

$$\bar{\mathcal{D}}_{EXT} := \{(z, \rho) \in \bar{S}_{EXT} : \rho \geq 0\} \quad (3.42)$$

and

$$\bar{\Sigma}_{ex} := \bar{\Sigma}_{EXT} \cap \hat{\Sigma}, \quad (3.43)$$

$$\bar{S}_{ex} := \{(z, \rho) \in \bar{S}_{EXT} : z + i\rho \in \bar{\Sigma}_{ex}\}, \quad (3.44)$$

$$\bar{\mathcal{D}}_{ex} := \{(z, \rho) \in \bar{S}_{ex} : \rho \geq 0\}. \quad (3.45)$$

There are four distinct cases:

- (i) Suppose that $\bar{\Sigma}_{ex} = \bar{\Sigma}_{EXT} - \{\infty\}$, but $\Pi_{44}(\tau)$ and $\tau^{-1}\Pi_{34}(\tau)$ have holomorphic extensions that cover $\tau = \infty$. The transformation $\mathbf{K}_4 \rightarrow \mathbf{K}_4 + k\mathbf{K}_3$, where k is a real number, induces $\Pi_{44}(\tau) \rightarrow \Pi_{44}(\tau)$ and $\Pi_{34}(\tau) \rightarrow \Pi_{34}(\tau) - 2k\tau$. So, this transformation can be used to make $\Pi(\tau)$ holomorphic at $\tau = \infty$, whereupon $\bar{\Sigma}_{ex} = \bar{\Sigma}_{EXT}$. Then we can and we do scale \mathbf{K}_4 and select $\chi(0)$ so that

$$\Pi(\infty) = I. \quad (3.46)$$

If it is also true that the matrix elements $\Pi_{ab}(\tau)$ satisfy

$$\tau\Pi_{34}(\tau)/\Pi_{44}(\tau) \rightarrow 0 \text{ as } \tau \rightarrow \infty, \quad (3.47)$$

then the Ernst potential \mathcal{E} of the SAV metric expressed as a function of Weyl canonical coordinates can be analytically extended to a domain that covers \bar{S}_{EXT} such that $\mathcal{E}(z, -\rho) = \mathcal{E}(z, \rho)$, and the restriction of \mathcal{E} to $\bar{\mathcal{D}}_{EXT}$ yields a SAV metric without singularities that is asymptotically flat and satisfies all of the requisite matching conditions at the zero pressure surface. So, in this case, we obtain an asymptotically flat global solution.

- (ii) Suppose that $\bar{\Sigma}_{ex} = \bar{\Sigma}_{EXT}$ as in the preceding case, but the condition (3.47) does not hold. Then, again, \mathcal{E} has an analytic continuation to a domain that covers \bar{S}_{EXT} such that $\mathcal{E}(z, -\rho) = \mathcal{E}(z, \rho)$, and the restriction of \mathcal{E} to $\bar{\mathcal{D}}_{EXT}$ yields a singularity free SAV metric that satisfies all of the requisite matching conditions. Again, we obtain a global solution that some relativists would regard as asymptotically flat. However, in this case, in a neighborhood of spatial infinity, though

$$\text{the limit } \nu_{NUT} \text{ of } \sqrt{z^2 + \rho^2} \chi(z, \rho) \text{ as } \sqrt{z^2 + \rho^2} \rightarrow \infty \quad (3.48)$$

exists, it is not zero. For this reason, the spacetime is not asymptotically flat in an orthodox sense. No example of this case is known, and it would be a shock if an example were found.

- (iii) Suppose that $\bar{\Sigma}_{ex}$ is a proper subspace of $\bar{\Sigma}_{EXT}$, and $\Pi(\tau)$ has no holomorphic extension to $\hat{\Sigma} \cup \Sigma_{EXT}$ (i.e., to a domain that covers $\bar{\Sigma}_{EXT}$), regardless of the choice of \mathbf{K}_4 . Then \mathcal{E} has an analytic extension to a domain which covers $\bar{\Sigma}_{ex}$ and satisfies $\mathcal{E}(z, -\rho) = \mathcal{E}(z, \rho)$. The restriction of \mathcal{E} to $\bar{\mathcal{D}}_{ex}$ yields a singularity-free SAV region that envelopes the fluid body and satisfies all requisite matching conditions. However, the full analytic continuation of $\mathcal{E}(z, \rho)$ will have at least one singularity on $\bar{\mathcal{D}}_{EXT}$ or at spatial infinity and will, therefore, not furnish a global solution without singularities. The Wahlquist solution may be in this case.
- (iv) Suppose that the holomorphic monodromy matrix $\Pi(\tau)$ does not exist [i.e., the conditions (ii) in Sec. 3C do not hold]. Then, either our matching formalism is not applicable to the given SAPF or there exists no SAV envelope that is free of singularities (including cusps) and that matches the given SAPF. When our matching formalism is applicable, the matching SAV envelope of the given SAPF is unique (in the sense that any two matching SAPF envelopes will have the same full analytic continuation). If our matching formalism is not applicable, then there may exist two or more matching SAPF envelopes with different full analytic continuations. Criteria that would tell us [before computing $\mathcal{Q}(\lambda, \tau)$] when our formalism is applicable remain to be discovered.

All of the conclusions that have been given above in (i) through (iv) follow from previous work by the authors.[3, 4, 5]

E Constructing Exterior Solution from $\Pi(\tau)$

If it is desired actually to construct the SAV solution that matches the given SAPF solution, one proceeds to identify a Kinnersley-Chitre transformation matrix $v(\tau)$ such that¹³

$$\Pi(\tau) = v(\tau)v(\tau)^T, \quad [v(\tau)]^* = v(\tau^*), \quad \text{and} \quad \det v(\tau) = 1. \quad (3.49)$$

Clearly, the matrix $v(\tau)$ is defined by Eq. (3.49) only up to a transformation

$$v(\tau) \rightarrow v(\tau)B(\tau), \quad (3.50)$$

where $B(\tau)$ is an orthogonal 2×2 matrix that satisfies $\det B(\tau) = 1$ and $B^*(\tau) = B(\tau)$. The choice of $B(\tau)$ has no effect upon the SAV solution. The axis values of the \mathcal{E} potential on the right side of the spinning spheroid are given by

$$\begin{aligned} \mathcal{E}(z, 0) &= \frac{1 + i\Pi_{34}(z)}{\Pi_{44}(z)} \\ &= \frac{v_{33}(z) + iv_{34}(z)}{-iv_{43}(z) + v_{44}(z)} \quad \text{for all } z \geq z(0), \end{aligned} \quad (3.51)$$

from which, as is well known, the SAV spacetime can be constructed by many modern methods that are based upon Riemann-Hilbert problems.

For example, our HHP corresponding to $v(\tau)$ can be solved in two successive steps[5], each of which involves well known mathematics. In the first step, which requires only that we compute a definite integral with a given integrand, Minkowski space is transformed into a Weyl static spacetime. In the second step, which requires that we solve an ordinary Fredholm equation of the second kind with a given kernel and a given inhomogeneous term, the Weyl static spacetime is transformed into the final SAV spacetime.

¹³Some choices of the factorization $\Pi(\tau) = v(\tau)v(\tau)^T$ may only be applicable to a domain $\hat{\Sigma} - \Sigma_{cut}$, where $\Sigma_{cut}^* = \Sigma_{cut}$ and is a union of cuts in $\hat{\Sigma}$. Each of these cuts crosses the real axis and its endpoints are (τ -independent) isolated branch points of $v(\tau)$. However, these branch points lead to no spacetime singularities, since the Ernst potential $\mathcal{E}(z, \rho)$ that is determined from $v(\tau)$ will have an analytic continuation that covers $\bar{\mathcal{D}}_{ex}$. In summary, there may be singularities of $v(\tau)$ that are not singularities of $\Pi(\tau)$; and it is only the singularities of $\Pi(\tau)$ that count.

4 Simple Illustrative Examples

We shall illustrate all stages of the Π -matrix method with examples. In this discussion we shall employ geometrical units; i.e., $G = 1$ and $c = 1$. Moreover, the time coordinate will be scale and the arbitrary constant in χ selected so that $\mathcal{Q}_0 := \mathcal{Q}(0, \tau) = I$ in Eqs. (3.25) and (3.33). In both cases the matching data $\rho(\lambda)$, $f(\lambda)$ and $\omega(\lambda)$ are obtained directly from the metric, while simple calculations yield the matching data $z(\lambda)$, $\chi(\lambda)$ and $\phi(\lambda)$.

A The Schwarzschild Interior Solution

The first example is the Schwarzschild interior solution[10],

$$ds^2 = \frac{dr^2}{1 - 2Mr^2/R^3} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) - e^{2\psi} dt^2, \quad (4.1)$$

where $0 \leq r \leq R$ and¹⁴

$$\psi(\mathbf{x}) = \ln \left[\frac{3}{2} - \frac{1}{2} k^{-1} (1 - 2Mr^2/R^3)^{1/2} \right], \quad k := (1 - 2M/R)^{1/2}. \quad (4.2)$$

In this chart, the pressure is given by

$$p(\mathbf{x}) = \frac{M}{(4\pi/3)R^3} \left(\frac{\sqrt{1 - 2Mr^2/R^3} - \sqrt{1 - 2M/R}}{3\sqrt{1 - 2M/R} - \sqrt{1 - 2Mr^2/R^3}} \right), \quad (4.3)$$

which is independent of θ and which vanishes when $r = R$.

Thus, for example, in the case of the Schwarzschild interior solution, the fields $P(\mathbf{x})$, $f(\mathbf{x})$, $\omega(\mathbf{x})$ and $\chi(\mathbf{x})$ are defined everywhere, and we find

$$P(\mathbf{x}) = re^{\psi(\mathbf{x})} \sin \theta, \quad f(\mathbf{x}) = e^{2\psi(\mathbf{x})}, \quad \omega(\mathbf{x}) = 0, \quad \chi(\mathbf{x}) = 0, \quad (4.4)$$

while

$$\begin{aligned} \star dP(\mathbf{x}) &= - \left[e^{\psi} (1 - 2Mr^2/R^3)^{1/2} + k^{-1} Mr^2/R^3 \right] r \sin \theta d\theta \\ &\quad + e^{\psi} (1 - 2Mr^2/R^3)^{-1/2} \cos \theta dr, \end{aligned} \quad (4.5)$$

$$\eta(\mathbf{x}) = -k^{-1} M (r/R)^3 \sin \theta d\theta, \quad (4.6)$$

¹⁴Note that we have scale the time coordinate t so that $\psi(\mathbf{x}) = 0$ at $r = R$.

where we have used the relations

$$\star dr = -\sqrt{r^2 - 2Mr^4/R^3}d\theta, \quad \star d\theta = dr/\sqrt{r^2 - 2Mr^4/R^3}. \quad (4.7)$$

On the zero pressure surface $r = R$, we obtain the matching data

Schwarzschild Interior Solution: $z(\lambda) = k^{-1}(R - M) \cos \lambda, \quad \rho(\lambda) = R \sin \lambda,$ $f(\lambda) = 1, \quad \omega(\lambda) = 0, \quad \chi(\lambda) = 0, \quad \phi(\lambda) = k^{-1}M \cos \lambda + \text{const.}$	(4.8)
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The value of the constant may be chosen at will. Since only $\dot{\phi}(\lambda)$ will be used, there is no need to be more specific.

Incidentally, in the case of the Schwarzschild interior solution, one can combine our expressions for $z(\lambda)$ and $\rho(\lambda)$ in the neat formula

$$z(\lambda) + i\rho(\lambda) = k^{-1}M \cosh(\beta + i\lambda), \quad \cosh \beta := (R - M)/M, \quad (4.9)$$

which has a holomorphic extension

$$z(\mathbf{x}) + i\rho(\mathbf{x}) = k^{-1}M \cosh(\beta + i\theta), \quad \cosh \beta := (r - M)/M, \quad (4.10)$$

where the fields $z(\mathbf{x})$ and $\rho(\mathbf{x})$ satisfy

$$dz(\mathbf{x}) = \star d\rho(\mathbf{x}) \text{ and } d\rho(\mathbf{x}) = -\star dz(\mathbf{x}). \quad (4.11)$$

While it is possible to employ a harmonic chart (z, ρ) to describe the solution, within the fluid the field $\rho(\mathbf{x})$ is *not* the same thing as the field $P(\mathbf{x})$, which is quite complicated when expressed in terms of z and ρ .

In the Weyl case, where $\omega(\mathbf{x}) = \chi(\mathbf{x}) = 0$, Eq. (3.22) reduces to

$$\frac{\partial \mathcal{Q}(\lambda, \tau)}{\partial \lambda} = -\frac{(\tau - z(\lambda))\dot{\psi}(\lambda) + \dot{\phi}(\lambda)}{\mu(\mathbf{x}(\lambda), \tau)} \sigma_3 \mathcal{Q}(\lambda, \tau). \quad (4.12)$$

Introducing $\Psi(\lambda, \tau)$ such that

$$\mathcal{Q}(\lambda, \tau) = e^{-\Psi(\lambda, \tau)\sigma_3}, \quad \Psi(0, \tau) = 0, \quad (4.13)$$

we may write

$$\frac{\partial \Psi(\lambda, \tau)}{\partial \lambda} = \frac{(\tau - z(\lambda))\dot{\psi}(\lambda) + \dot{\phi}(\lambda)}{\mu(\mathbf{x}(\lambda), \tau)}. \quad (4.14)$$

Thus, in the case of the Schwarzschild interior solution, we have

$$\frac{\partial \Psi(\lambda, \tau)}{\partial \lambda} = \frac{-k^{-1} M \sin \lambda}{[(\tau - z(\lambda))^2 + \rho(\lambda)^2]^{1/2}}, \quad (4.15)$$

and hence,

$$\Psi(\lambda, \tau) = \ln \left[\frac{\left(\frac{R-M}{M}\right) \tau - \left(\frac{M}{R-M}\right) z(\lambda) - \mu(\mathbf{x}(\lambda), \tau)}{\left(\frac{R-2M}{M}\right) \left(\tau + \left(\frac{M}{R-M}\right) z(0)\right)} \right]. \quad (4.16)$$

The function $\mathcal{Q}(\lambda, \tau)$ is given by Eq. (4.13), and

$$\mathcal{Q}'(\lambda, \tau) = e^{-\Psi'(\lambda, \tau) \sigma_3}, \quad (4.17)$$

where

$$\Psi'(\lambda, \tau) = \ln \left[\frac{\left(\frac{R-M}{M}\right) \tau - \left(\frac{M}{R-M}\right) z(\lambda) + \mu(\mathbf{x}(\lambda), \tau)}{\left(\frac{R-2M}{M}\right) \left(\tau + \left(\frac{M}{R-M}\right) z(0)\right)} \right]. \quad (4.18)$$

Hence

$$\Pi(\tau) := -J \mathcal{Q}'(0, \tau) J = e^{2\xi(\tau) \sigma_3}, \quad (4.19)$$

where

$$\xi(\tau) := -\ln k + \frac{1}{2} \ln \left(\frac{k\tau - M}{k\tau + M} \right). \quad (4.20)$$

Therefore, we can select the K-C transformation matrix

$$v(\tau) = e^{\xi(\tau) \sigma_3} = \begin{pmatrix} k^{-1} & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} \left(\frac{k\tau - M}{k\tau + M}\right)^{\frac{1}{2}} & 0 \\ 0 & \left(\frac{k\tau + M}{k\tau - M}\right)^{\frac{1}{2}} \end{pmatrix}. \quad (4.21)$$

By Eq. (3.51) the axial values of the \mathcal{E} -potential are given by

$$k^2 \mathcal{E}(z, 0) = \frac{kz - M}{kz + M}, \quad kz \geq R - M. \quad (4.22)$$

The factors k^2 and k can be suppressed by rescaling the time coordinate t and the spectral parameter τ . We then obtain the well known axis values of the exterior Schwarzschild solution. The values of $\mathcal{E}(z, \rho)$ for $\rho > 0$ are obtained by solving the authors' HHP or one of the other well known methods.

B A Winicour Dust Solution

A somewhat more complicated example is provided by a Winicour dust metric[11]

$$ds^2 = e^{-b^2\rho^2/4}(dz^2 + d\rho^2) + \rho^2 d\varphi^2 - [dt + (b\rho^2/2)d\varphi]^2, \quad (4.23)$$

for which the pressure vanishes everywhere, while the energy density is given by¹⁵

$$\epsilon(z, \rho) = \frac{1}{8\pi} b^2 e^{b^2\rho^2/4}. \quad (4.24)$$

Because the pressure vanishes everywhere, one can select any convenient matching surface; for example,

$$z = R \cos \lambda, \quad \rho = R \sin \lambda, \quad (4.25)$$

where $0 \leq \lambda \leq \pi$. Note that the requirement $g_{33} > 0$ yields $(bR/2)^2 < 1$.

In the case of the dust metric, $Z(\mathbf{x})$, $P(\mathbf{x})$, $f(\mathbf{x})$, $\omega(\mathbf{x})$ and $\chi(\mathbf{x})$ are defined throughout the chart, and we find

$$Z(\mathbf{x}) = z, \quad P(\mathbf{x}) = \rho, \quad f(\mathbf{x}) = 1, \quad \omega(\mathbf{x}) = -\frac{1}{2}b\rho^2, \quad \chi(\mathbf{x}) = b(z - R), \quad (4.26)$$

while

$$\eta = \frac{1}{4}b^2\rho^2 dz. \quad (4.27)$$

On the selected zero pressure surface (4.25), we obtain the matching data

Winicour Dust Metric:

$$\begin{aligned} z(\lambda) &= R \cos \lambda, & \rho(\lambda) &= R \sin \lambda, \\ f(\lambda) &= 1, & \omega(\lambda) &= -\frac{1}{2}bR^2 \sin^2 \lambda, \\ \chi(\lambda) &= bR(\cos \lambda - 1), & \phi(\lambda) &= \frac{1}{4}b^2R^2 \left(\cos \lambda - \frac{1}{3} \cos^3 \lambda - \frac{2}{3} \right). \end{aligned}$$

(4.28)

When $f(\lambda)$ is independent of λ , it is obvious from Eqs. (2.13) and (3.21) that those terms in $\Delta(\mathbf{x}, \tau)$ that are proportional to the matrix σ_3 all vanish.¹⁶ We find that Eq. (3.22) assumes the form

$$\begin{aligned} \frac{\partial \mathcal{Q}(\lambda, \tau)}{\partial \lambda} &= \frac{b}{2} R \sin \lambda \left[\frac{\tau}{m(\lambda, \tau)} \sigma_1 + J \right] \mathcal{Q}(\lambda, \tau), \\ \mathcal{Q}(0, \tau) &= I, \end{aligned} \quad (4.29)$$

¹⁵We employ units such that $c = G = 1$.

¹⁶For this reason it is not really necessary to evaluate $\phi(\lambda)$.

where

$$m(\lambda, \tau) := [\tau^2 - 2\tau R \cos \lambda + R^2]^{1/2}. \quad (4.30)$$

Consider the following differential equation in the complex plane:¹⁷

$$\frac{\partial \mathcal{F}(\zeta, \tau)}{\partial \zeta} = \frac{b}{2} \left(\sigma_1 + \frac{\zeta}{\tau} J \right) \mathcal{F}(\zeta, \tau) \quad (4.31)$$

such that

$$\text{dom } \mathcal{F} = \{(\zeta, \tau) \in C^2 : \zeta \neq \infty \text{ and } \tau \notin \{0, \infty\}\} \quad (4.32)$$

and

$$\mathcal{F}(\tau - R, \tau) = I. \quad (4.33)$$

If \mathcal{F} is the function defined by the above Eqs. (4.31) to (4.33), then it is easy to prove with the aid of

$$\frac{\partial m(\lambda, \tau)}{\partial \lambda} = R \sin \lambda \frac{\tau}{m(\lambda, \tau)} \quad (4.34)$$

and

$$m(0, \tau) = \tau - R \quad (4.35)$$

that

$$\mathcal{Q}(\lambda, \tau) := \mathcal{F}(m(\lambda, \tau), \tau) \quad (4.36)$$

is the solution of Eqs. (4.29) over the domain

$$\{(\lambda, \tau) : 0 \leq \lambda \leq \pi \text{ and } \tau \in C - \bar{K}(\mathbf{x}(\lambda)) - \{0, \infty\}\}. \quad (4.37)$$

The triad of equations (4.31), (4.32) and (4.33) is collectively equivalent to Eq. (4.32) taken together with the following integral equation:

$$\mathcal{F}(\zeta, \tau) = I + \frac{b}{2} \int_{\tau-R}^{\zeta} d\zeta' \left(\sigma_1 + \frac{\zeta'}{\tau} J \right) \mathcal{F}(\zeta', \tau) \quad (4.38)$$

or, equivalently,

$$\mathcal{G}(\eta, \tau) = I + \frac{b}{2} \int_0^{\eta} d\eta' \left(\gamma(\tau) + J \frac{\eta'}{\tau} \right) \mathcal{G}(\eta', \tau), \quad (4.39)$$

¹⁷ For fixed τ , each matrix element of \mathcal{F} satisfies the fourth order equation $\square^2 \mathcal{F} + \beta^2 \mathcal{F} = 0$ with $\square := \partial^2 / \partial \zeta^2 - \beta^2 (\tau^2 - \zeta^2)$ and $\beta := b/(2\tau)$. We do not know if this differential equation or the second order equations $\square \mathcal{F} \pm i\beta \mathcal{F} = 0$ have been the subject of study elsewhere.

where

$$\gamma(\tau) := \sigma_1 + J \left(\frac{\tau - R}{\tau} \right) \quad (4.40)$$

and

$$\eta := \zeta - (\tau - R) \text{ and } \mathcal{G}(\eta, \tau) := \mathcal{F}(\eta + \tau - R, \tau). \quad (4.41)$$

The solution of Eq. (4.39) is given by the infinite series

$$\mathcal{G}(\eta, \tau) = \sum_{n=0}^{\infty} \left(\frac{b}{2} \right)^n \mathcal{G}_n(\eta, \tau), \quad (4.42)$$

where

$$\mathcal{G}_0(\eta, \tau) = I \quad (4.43)$$

and, for all $n \geq 0$,

$$\mathcal{G}_{n+1}(\eta, \tau) = \int_0^\eta d\eta' \left[\gamma(\tau) + J \frac{\eta'}{\tau} \right] \mathcal{G}_n(\eta', \tau). \quad (4.44)$$

By mathematical induction one proves that

$$\mathcal{G}_n(\eta, \tau) = \frac{\eta^n}{n!} g_n(\eta, \tau), \quad (4.45)$$

where

$$g_n(\eta, \tau) = \sum_{k=0}^n g_{nk}(\tau) (\eta/\tau)^k \quad (4.46)$$

and the coefficients $g_{nk}(\tau)$ are to be computed from the recursion relation

$$g_{n+1,k}(\tau) = \left(\frac{n+1}{n+1+k} \right) [\gamma(\tau) g_{nk}(\tau) + J g_{n,k-1}(\tau)] \quad (4.47)$$

and the conditions

$$g_{00}(\tau) = I, \quad g_{nk}(\tau) = 0 \text{ if } k < 0 \text{ and if } k > n. \quad (4.48)$$

Thus, $g_0(\eta, \tau) = I$,

$$g_1(\eta, \tau) = \gamma(\tau) + J \frac{\eta}{2\tau} \quad (4.49)$$

and

$$g_2(\eta, \tau) = \gamma(\tau)^2 + \left(J\gamma(\tau) + \frac{1}{2}\gamma(\tau)J \right) \frac{2\eta}{3\tau} - I \frac{\eta^2}{4\tau^2}, \quad (4.50)$$

where

$$\gamma(\tau)^2 = I \left[1 - \left(\frac{\tau - R}{\tau} \right)^2 \right] \quad (4.51)$$

and, since $J\sigma_1 = \sigma_3$,

$$J\gamma(\tau) + \frac{1}{2}\gamma(\tau)J = \frac{1}{2}\sigma_3 - \frac{3}{2}I \left(\frac{\tau - R}{\tau} \right). \quad (4.52)$$

One further proves by mathematical induction that

$$g_{nk}(\tau) \text{ is a holomorphic function of } \tau \text{ throughout } C - \{0\} \quad (4.53)$$

and

$$\begin{aligned} \tau^n g_{nk}(\tau) &\text{ is a holomorphic function of } \tau \\ &\text{ throughout } C - \{\infty\} \text{ and has the value} \\ &(-JR)^n \delta_{k0} \text{ at } \tau = 0. \end{aligned} \quad (4.54)$$

From Eqs. (4.36) and (4.41),

$$\mathcal{Q}(\lambda, \tau) = \mathcal{F}(m(\lambda, \tau), \tau) = \mathcal{G}(m(\lambda, \tau) - (\tau - R), \tau) \quad (4.55)$$

for all (λ, τ) in the domain (4.37). From Eqs. (4.42), (4.45), (4.46), (4.53), (4.54) and the existence of the limits

$$\lim_{\tau \rightarrow \infty} [m(\lambda, \tau) - (\tau - R)] = R(1 - \cos \lambda) \quad (4.56)$$

and

$$\lim_{\tau \rightarrow 0} \left[\frac{m(\lambda, \tau) - (\tau - R)}{\tau} \right] = -(1 - \cos \lambda), \quad (4.57)$$

it follows as expected that

$$\begin{aligned} &\text{for each } \lambda \in [0, \pi], \mathcal{F}(m(\lambda, \tau), \tau) \text{ has a} \\ &\text{holomorphic extension to all of} \\ &C - \bar{K}(\mathbf{x}(\lambda)), \end{aligned} \quad (4.58)$$

and the equality (4.55) holds for all (λ, τ) in the domain

$$\text{dom } \mathcal{Q}(\lambda, \tau) := \{(\lambda, \tau) : 0 \leq \lambda \leq \pi \text{ and } \tau \in C - \bar{K}(\mathbf{x}(\lambda))\}. \quad (4.59)$$

Incidentally, the reader can use Eqs. (4.56) and (4.57) to compute

$$\begin{aligned}\mathcal{Q}(\lambda, \infty) &= \exp \left[\frac{b}{2} R(1 - \cos \lambda)(\sigma_1 + J) \right] \\ &= I + bR(1 - \cos \lambda) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\end{aligned}\tag{4.60}$$

$$\mathcal{Q}(\lambda, 0) = \exp \left[\frac{b}{2} R(1 - \cos \lambda)J \right].\tag{4.61}$$

Since $\mathcal{Q}_0 := \mathcal{Q}(0, \tau) = I$ in Eq. (3.33), the Pi -matrix is given by

$$\Pi(\tau) = -J\mathcal{G}(-2(\tau - R), \tau)J.\tag{4.62}$$

So, from Eqs. (4.42), (4.45) and (4.46),

$$\Pi(\tau) = -J \sum_{n=0}^{\infty} \frac{[-b\tau(1 - R/\tau)]^n}{n!} g_n(-2(\tau - R), \tau)J\tag{4.63}$$

and

$$g_n(-2(\tau - R), \tau) = \sum_{k=0}^n g_{nk}(\tau) [-2(1 - R/\tau)]^k.\tag{4.64}$$

Equations (4.49) and (4.50) imply that $g_1(-2(\tau - R), \tau)$ and $g_2(-2(\tau - R), \tau)$ are not zero at $\tau = \infty$. Further calculations reveal that $g_n(-2(\tau - R), \tau)$ is not zero at $\tau = \infty$ for all $n \leq 10$. Granting that this continues to be true for all values of n , we may conclude that $\Pi(\tau)$ has an isolated essential singularity at $\tau = \infty$, and, therefore, the SAV that matches the dust metric at the boundary we have been considering is not asymptotically flat. A proof of this conclusion, however likely it seems, remains to be found.

5 Generalizations

The Π -matrix method can be generalized in a number of respects. The stationary axisymmetric source need not be of spheroidal shape. Toroidal sources, or multiple spheroidal sources spinning on a common axis, would be interesting. The source need not even be a perfect fluid. Stationary axisymmetric charge and current density may be involved, with resulting stationary axisymmetric electromagnetic fields. In principle, all such “joining

problems” can be handled by an extended Π -matrix approach. Once the axis values of the the complex potentials \mathcal{E} and Φ have been deduced from $\Pi(\tau)$, the exterior electrovac fields can be constructed by solving the electrovac version of the authors’ HHP. In this case the calculational methods devised by Alekseev[12] and Sibgatullin[13] are germane.

We have in this first exposition of the Π -matrix method avoided the mathematical complications that such generalizations would entail, hoping that this would make it easier for the reader to appreciate the general idea behind this approach.

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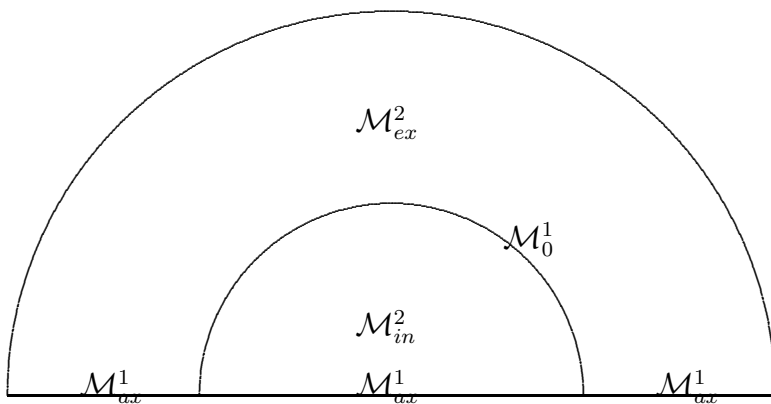


Figure 1: Subspaces of the topological space \mathcal{M}^2